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An Efficient Method for Solving Systems of Fractional Integro-Differential Equations

S. MOMANI AND R. QARALLEH
Department of Mathematics, Mutah University
P.O. Box 7, Al-Karak, Jordan
shahermm@yahoo.com

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Abstract—In this study, a decomposition method for approximating the solution of systems of fractional integro-differential equations is implemented. The fractional derivative is considered in the Caputo sense. The approximate solutions are calculated in the form of a convergent series with easily computable components. Numerical results show that this approach is easy to implement and accurate when applied to integro-differential equations. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In recent years, the Adomian decomposition method (ADM) [1,2] has been applied to a wide class of stochastic and deterministic problems in many areas of mathematics and physics. This computational method yields analytical solutions and has certain advantages over standard numerical methods. It is free from roundoff errors as it does not involve discretization, and does not require large computer memory or power. Adomian gave a review of the decomposition method in [1].

The ADM handles both linear and nonlinear problems [3]. For nonlinear problems, the ADM is extremely efficient in supplying analytical approximations which converge very rapidly. It solves these types of problems without requiring linearization, perturbation, or unjustified assumptions which may change the problem being solved. Recently [4,5], the solution of fractional ordinary differential equations has been obtained through the Adomian decomposition method. The application of the Adomian decomposition method for the solution of fractional partial differential equations has also been established in [6–8].

Fractional derivatives arise in many physical and engineering problems such as frequency dependent damping behavior of materials, diffusion processes, motion of a large thin plate in a

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Newtonian fluid, creeping and relaxation functions for viscoelastic materials. A brief review of fraction calculus is given in [9,10].

In this paper, we are concerned with providing a numerical scheme for the solution of a system of fractional integro-differential equations of the general form

$$\frac{d^\alpha y_i}{dt^\alpha} = f_i\left(t, y_1, y_2, \ldots, y_n, \int_0^t K_i(t, y_1, y_2, \ldots, y_n)\,dx\right), \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

subject to the initial conditions

$$y_i^k(0) = c_{ik}, \qquad k = 0, 1, \ldots, m-1, \quad m-1 < \alpha \leqslant m, \quad m \in N, \tag{1.2}$$

where c_{ik} is a specified constant vector, α is a parameter describing the order of the time-fractional derivative, and $y_i(t)$, the solution vector, is assumed to be a causal function of time, i.e., vanishing for $t < 0$. The fractional derivative is considered in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. When $\alpha = 1$, the fractional system of equations (1.1) reduces to the classical system of integro-differential equations.

The paper is organized as follows. A brief review of the fractional calculus theory is given in Section 2. In Section 3, we use the decomposition method to construct our numerical solutions for a system of fractional integro-differential equations. In Section 4, we present four examples to show the efficiency and simplicity of the method.

2. BASIC DEFINITIONS

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

DEFINITION 2.1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number p $(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.*

DEFINITION 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \qquad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in [11–13]. We mention only the following.

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$:

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
(3) $J^\alpha x^\gamma = \big(\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)\big) x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [14].

DEFINITION 2.3. *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$D_*^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{2.1}$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

LEMMA 2.1. *If $m-1 < \alpha \leq m$, $m \in N$, and $f \in C_{\mu}^m$, $\mu \geq -1$, then*

$$D_*^{\alpha} J^{\alpha} f(x) = f(x)$$

and

$$J^{\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x > 0.$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem [9]. For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

3. ANALYSIS OF THE NUMERICAL METHOD

The system of equations (1.1) can be written in terms of operator form as

$$D_*^{\alpha} y_i = f_i \left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\, dx \right), \qquad i = 1, 2, \dots, n, \tag{3.1}$$

where the fractional differential operator D_*^{α} is defined as in equation (2.1) denoted by

$$D_*^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}}.$$

Applying the operator J^{α}, the inverse of the operator D_*^{α}, to both sides of equation (3.1) yields

$$y_i(t) = \sum_{k=0}^{m-1} y_i^{(k)}(0^+) \frac{t^k}{k!} + J^{\alpha} \left(f_i \left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\, dx \right) \right). \tag{3.2}$$

The Adomian decomposition method [1,2] suggests the solution $y_i(t)$ be decomposed into the infinite series of components

$$y_i(t) = \sum_{j=0}^{\infty} f_{i,j} \left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\, dx \right), \tag{3.3}$$

substituting the initial conditions (1.2) into (3.2) and identify the zeroth component $y_{i,0}$ by the term arising from the initial condition and from the source term, then we have the following recursive relations:

$$\begin{aligned} y_{i,0}(t) &= \sum_{k=0}^{m-1} y_i^{(k)}(0^+) \frac{t^k}{k!}, && i = 1, 2, \dots, n, \\ y_{i,j+1}(t) &= J^{\alpha} \left(f_{i,j} \left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\, dx \right) \right), && j = 0, 1, \dots, \end{aligned} \tag{3.4}$$

where $m-1 < \alpha \le m$. The components of $y_{i,j}(t)$, $j \ge 1$ are determined in the following recursive way:

$$\begin{aligned} y_{i,1}(t) &= J^{\alpha}\left(f_{i,0}\left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\,dx\right)\right), \\ y_{i,2}(t) &= J^{\alpha}\left(f_{i,1}\left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\,dx\right)\right), \\ y_{i,3}(t) &= J^{\alpha}\left(f_{i,2}\left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\,dx\right)\right), \\ &\vdots \end{aligned} \tag{3.5}$$

As a result, the series solution is given by

$$y_i(t) = y_{i,0}(t) + \sum_{j=1}^{\infty}\left[J^{\alpha}\left(f_{i,j}\left(t, y_1, y_2, \dots, y_n, \int_0^t K_i(t, y_1, y_2, \dots, y_n)\,dx\right)\right)\right]. \tag{3.6}$$

Finally, we approximate the solution $y_i(t)$ by the truncated series

$$\phi_{i,N}(t) = \sum_{j=0}^{N-1} y_{i,j}(t) \qquad \text{and} \qquad \lim_{N\to\infty} \phi_{i,N}(t) = y_i(t). \tag{3.7}$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions generally converge very rapidly. The convergence of the decomposition series has been investigated in [15,16].

4. ILLUSTRATIVE EXAMPLES

The decomposition method provides a reliable technique that requires less work if compared with traditional techniques. To give a clear overview of the methodology, the following four examples will be discussed. All the results are calculated by using the symbolic calculus software Mathematica.

EXAMPLE 4.1. Consider the following fractional linear system of fractional integro-differential equations:

$$\begin{aligned} D_*^{\alpha} y_1 &= 1 + t + t^2 - y_2 - \int_0^t (y_1(x) + y_2(x))\,dx, \qquad 0 < \alpha \leqslant 1, \\ D_*^{\alpha} y_2 &= -1 - t + y_1 - \int_0^t (y_1(x) - y_2(x))\,dx, \end{aligned} \tag{4.1}$$

subject to the initial conditions

$$y_1(0) = 1, \qquad y_2(0) = -1. \tag{4.2}$$

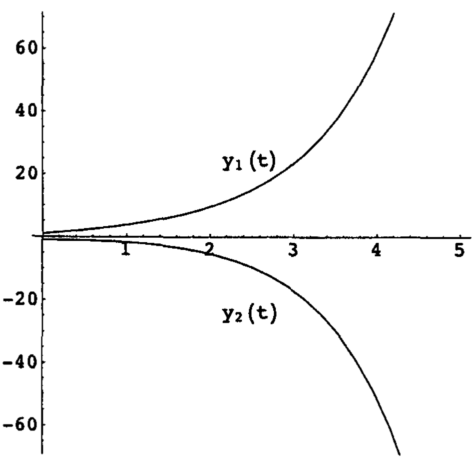
In order to solve this system by using the decomposition method, we simply substitute the initial condition (4.2) into equation (3.4) to obtain the following recursive relations:

$$\begin{aligned} y_{1,0} &= 1 + J^{\alpha}\left[1 + t + t^2\right], \\ y_{2,0} &= -1 + J^{\alpha}[-1 - t], \\ y_{1,j+1} &= J^{\alpha}\left[-y_{2,j} - \int_0^t (y_{1,j}(x) + y_{2,j}(x))\,dx\right], \\ y_{2,j+1} &= J^{\alpha}\left[y_{1,j} - \int_0^t (y_{1,j}(x) - y_{2,j}(x))\,dx\right]. \end{aligned}$$

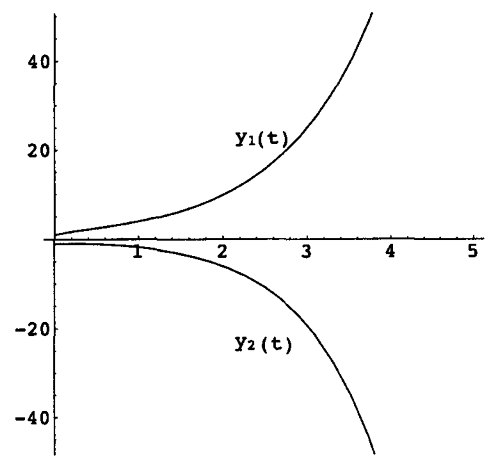
Using the above recursive relationship and Mathematica, the first four terms of the decomposition series are given by

$$
\begin{aligned}
y_1 &= 1 + \frac{2}{\Gamma(1+\alpha)}t^{\alpha} + \frac{1}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{2}{(2+3\alpha+\alpha^2)\Gamma(1+\alpha)}t^{2+\alpha} \\
&+ \frac{4^{-\alpha}\sqrt{\pi}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)}t^{2\alpha} + \frac{1}{\Gamma(2+2\alpha)}t^{1+2\alpha} - \frac{2}{\Gamma(4+2\alpha)}t^{3+2\alpha} + \cdots \\
y_2 &= -1 - \frac{3}{\Gamma(2+\alpha)}t^{1+\alpha} + \frac{1}{\Gamma(1+2\alpha)}t^{2\alpha} - \frac{1}{\Gamma(2+2\alpha)}t^{1+2\alpha} \\
&- \frac{1}{\Gamma(3+2\alpha)}t^{2+2\alpha} - \frac{2}{\Gamma(4+2\alpha)}t^{3+2\alpha} + \cdots
\end{aligned}
\tag{4.3}
$$

Figures 1a and 1b show the evolution results for the fractional integro-differential equation (4.1) when $\alpha = 1$ and $\alpha = 0.75$, respectively. The value $\alpha = 1$ is the only case for which we know the exact solution ($y_1 = t + e^t$, $y_1 = t - e^t$) and our approximate solution is in good agreement with the exact values and the approximate solution obtained by Biazar [17]. From Figures 1a and 1b, it is easy to conclude that the solution continuously depends on the time-fractional derivative.



(a) $\alpha = 1$.



(b) $\alpha = 0.75$.

Figure 1. Plots of system (4.1).

EXAMPLE 4.2. Consider the following nonlinear system of fractional differential equations:

$$
\begin{aligned}
D_*^{\alpha}x(t) &= x(t)(a - by(t)), \qquad 0 < \alpha \leqslant 1, \\
D_*^{\alpha}y(t) &= -y(t)(c - dx(t)),
\end{aligned}
\tag{4.4}
$$

subject to the initial conditions

$$
x(0) = x_0, \qquad y(0) = y_0. \tag{4.5}
$$

Applying J^{α}, the inverse operator of D_*^{α}, to both sides of the system of equations (4.4), we obtain the following system of fractional integral equations:

$$
\begin{aligned}
x(t) &= x(0) + J^{\alpha}\left[ax(t) - bx(t)y(t)\right], \\
y(t) &= y(0) - J^{\alpha}\left[cy(t) - dx(t)y(t)\right].
\end{aligned}
\tag{4.6}
$$

In order to solve the above system and in addition to equation (3.4), we define the nonlinear term by $\Phi(x,y) = xy = \sum_{j=0}^{\infty} A_j(t)$, and where A_j are the appropriate Adomian polynomials generated for the specific nonlinearities in equation (4.6), a definition for A_j is given as

$$A_n(x_0,\dots,x_n;y_0,\dots,y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\Phi \sum_{k=0}^{n} \lambda x_k, \sum_{k=0}^{n} \lambda y_k\right]_{\lambda=0}, \qquad j \geq 0. \tag{4.7}$$

This formula is easy to compute by using Mathematica software or by writing a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. The first few terms of Adomian polynomials for the nonlinear term are:

$$\begin{aligned} A_0 &= x_0(t)y_0(t), \\ A_1 &= x_1(t)y_0(t) + x_0(t)y_1(t), \\ A_2 &= x_2(t)y_0(t) + x_1(t)y_1(t) + x_0(t)y_2(t), \\ &\vdots \end{aligned} \tag{4.8}$$

Substituting initial conditions (4.5) into equation (4.6) and using equation (4.7) to calculate the Adomian polynomials, yields the following recursive relations:

$$\begin{aligned} x_0(t) &= x_0, \\ y_0(t) &= y_0, \\ x_{j+1}(t) &= J^{\alpha}\left(ax_j(t) - bA_j\right), \\ y_{j+1}(t) &= -J^{\alpha}\left(cy_j(t) - dA_j\right). \end{aligned} \tag{4.9}$$

Using the above recursive relationship and Mathematica, the first three terms of the decomposition series are given by

$$\begin{aligned} x(t) &= x_0 + \frac{x_0(a - by_0)}{\Gamma(1+\alpha)}t^{\alpha} \\ &\quad + \frac{4^{-\alpha}x_0\left(a^2 - 2aby_0 + by_0(c - dx_0 + by_0)\right)\cos(\pi\alpha)\Gamma(1/2-\alpha)}{\sqrt{\pi}\,\Gamma(1+\alpha)}t^{2\alpha} + \cdots \\ y(t) &= y_0 - \frac{(c - dx_0)y_0}{\Gamma(1+\alpha)}t^{\alpha} \\ &\quad + \frac{4^{-\alpha}y_0\left(c^2 - 2cdx_0 + dx_0(a + dx_0 - by_0)\right)\cos(\pi\alpha)\Gamma(1/2-\alpha)}{\sqrt{\pi}\,\Gamma(1+\alpha)}t^{2\alpha} + \cdots. \end{aligned} \tag{4.10}$$

For numerical study, the values used are shown in Table 1.

Table 1.

Case	x_0	y_0	a	b	c	d
1	14	18	1	1	0.1	1
2	14	18	0.1	1	1	1
3	16	10	0.1	1	1	1
4	16	10	1	1	0.1	1

According to the values introduced in the table, the following solutions are derived.

CASE 1.

$$x(t) = 14 - \frac{238t^{\alpha}}{\Gamma(1+\alpha)} + \frac{962.797\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$
$$y(t) = 18 + \frac{250.2t^{\alpha}}{\Gamma(1+\alpha)} - \frac{1428.99\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$

CASE 2.

$$x(t) = 14 - \frac{250.6t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2144.21\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$
$$y(t) = 18 + \frac{234t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2603.38\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$

CASE 3.

$$x(t) = 16 - \frac{158.4t^{\alpha}}{\Gamma(1+\alpha)} - \frac{1474.4\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$
$$y(t) = 10 + \frac{150t^{\alpha}}{\Gamma(1+\alpha)} + \frac{1180.45\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$

CASE 4.

$$x(t) = 16 - \frac{144t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2212.02\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots,$$
$$y(t) = 10 + \frac{159t^{\alpha}}{\Gamma(1+\alpha)} + \frac{1928.61\, 4^{-\alpha}t^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1/2+\alpha)} + \cdots.$$

Setting $\alpha = 1$, we obtain the solution obtained by Biazar and Montazeri [18] which corresponds to a system of ordinary differential equations. To examine the effect of varying the order of the fractional derivative on the behavior of solution, we take $\alpha = 1$ and 0.75. Figures 2–5 show the approximate solutions for Cases 1, 2, 3, and 4, respectively. It can be seen from Figures 2–5 that the solution continuously depends on the fractional derivative.

EXAMPLE 4.3. Consider the following system of nonlinear fractional differential equations:

$$\begin{aligned} D_*^{\alpha} y_1 &= 2y_2^2, && 0 < \alpha \leqslant 1, \\ D_*^{\beta} y_2 &= t y_1, && 0 < \beta \leqslant 1, \\ D_*^{\gamma} y_3 &= y_2 y_3, && 0 < \gamma \leqslant 1, \end{aligned} \tag{4.11}$$

subject to the initial conditions

$$y_1(0) = 0, \qquad y_2(0) = 1, \qquad y_3(0) = 1, \tag{4.12}$$

where D_*^{α}, D_*^{β}, and D_*^{γ} denote Caputo fractional derivatives of order α, β, and γ, respectively.

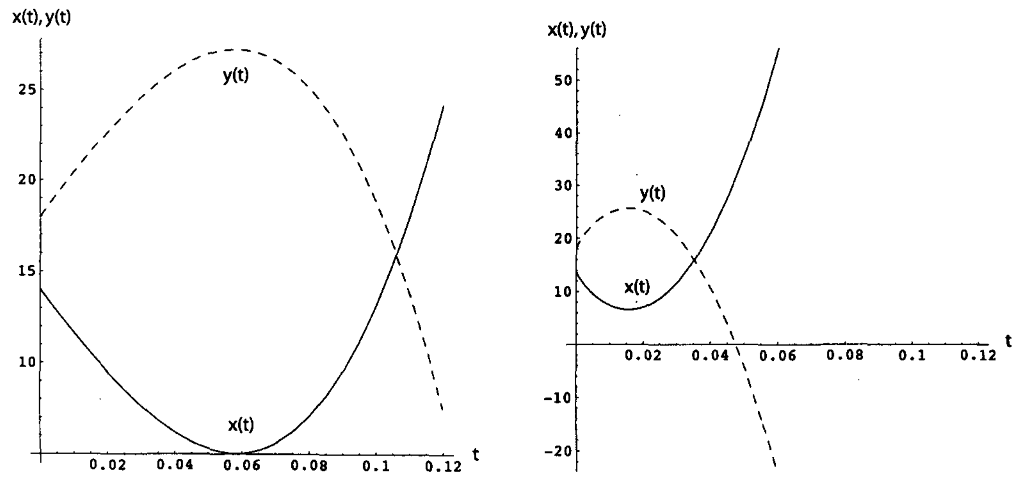
Following the analysis presented above gives the recurrence relation

$$\begin{aligned} y_{1,0} &= 0, & y_{1,j+1} &= 2J^{\alpha}(B_j), \\ y_{2,0} &= 1, & y_{2,j+1} &= J^{\beta}(t y_1), \\ y_{3,0} &= 1, & y_{3,j+1} &= J^{\gamma}(C_j), \end{aligned}$$

where $y_2^2 = \sum_{j=0}^{\infty} B_j$, $y_2 y_3 = \sum_{j=0}^{\infty} C_j$, and the B_j and C_j are the appropriate Adomian polynomials generated for the specific nonlinearity in this equation.

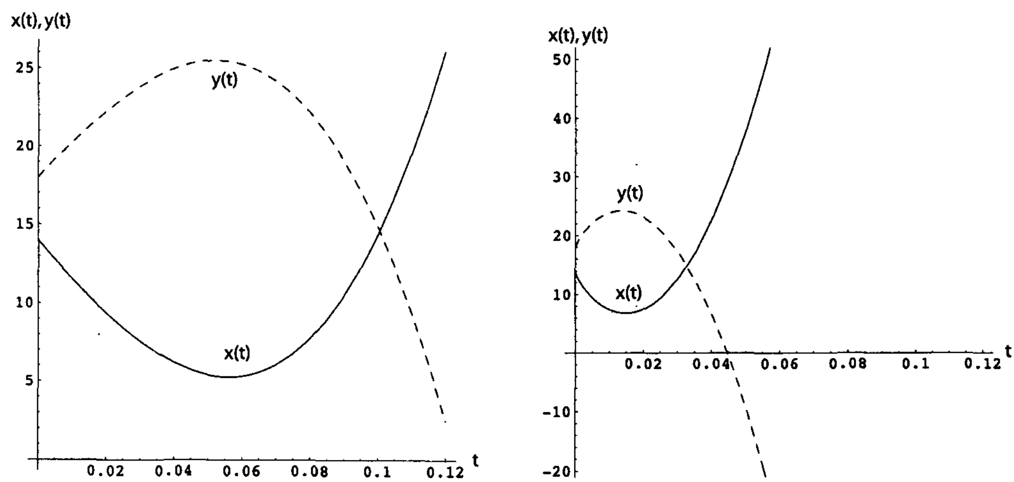
This relation enables us to determine the first few components as follows:

$$\begin{aligned} y_{1,0} &= 0, & y_{1,1} &= \frac{2t^{\alpha}}{\Gamma(1+\alpha)}, & y_{1,2} &= 0, \\ y_{2,0} &= 1, & y_{2,1} &= 0, & y_{2,2} &= \frac{2(1+\alpha)t^{1+\alpha+\beta}}{\Gamma(2+\alpha+\beta)}, \\ y_{3,0} &= 1, & y_{3,1} &= \frac{t^{\gamma}}{\Gamma(1+\gamma)}, & y_{3,2} &= \frac{4^{-\gamma}\sqrt{\pi}\, t^{2\gamma}}{\Gamma(1+\gamma)\Gamma(1/2+\gamma)}, \end{aligned}$$



(a) $\alpha = 1$. (b) $\alpha = 0.75$.

Figure 2. Plots of system (4.4) for Case 1.



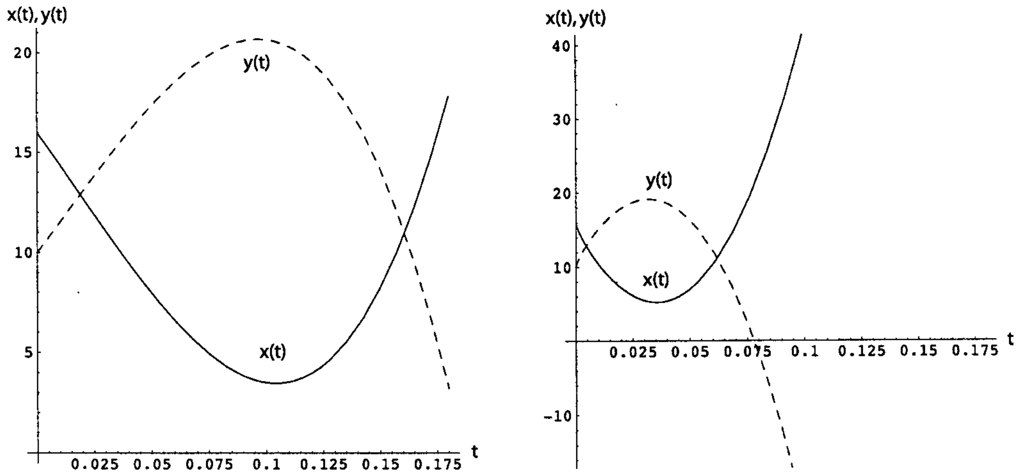
(a) $\alpha = 1$. (b) $\alpha = 0.75$.

Figure 3. Plots of system (4.4) for Case 2.

and so on. The solution in a series form is given by

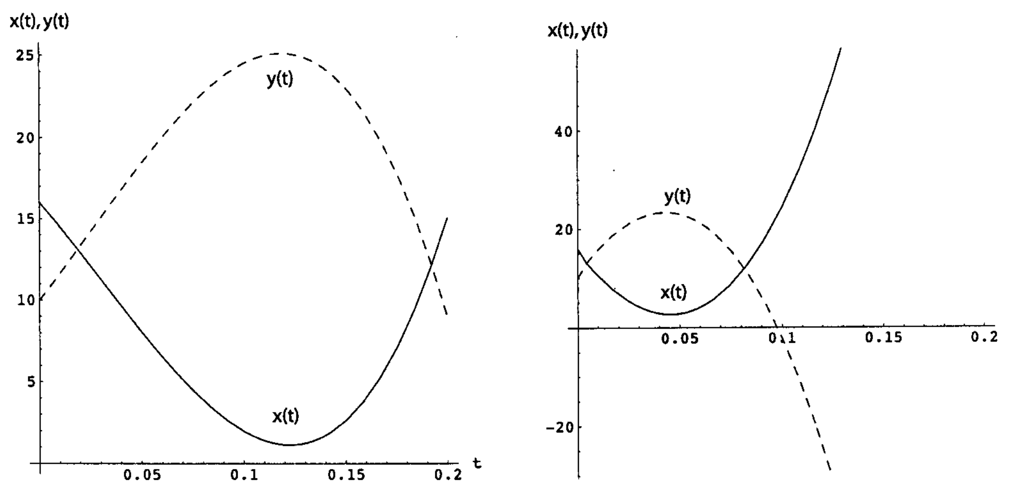
$$\begin{aligned} y_1 &= \frac{2t^{\alpha}}{\Gamma(1+\alpha)} + \cdots, \\ y_2 &= 1 + \frac{2(1+\alpha)t^{1+\alpha+\beta}}{\Gamma(2+\alpha+\beta)} + \cdots, \\ y_3 &= 1 + \frac{t^{\gamma}}{\Gamma(1+\gamma)} + \frac{4^{-\gamma}\sqrt{\pi}\, t^{2\gamma}}{\Gamma(1+\gamma)\Gamma(1/2+\gamma)} + \cdots. \end{aligned} \tag{4.13}$$

In particular, when $(\alpha, \beta, \gamma)^t = (0.5, 0.4, 0.3)^t$, solution (4.13) reduces to the solution obtained in [19] by using an iterative method. In Figures 6a and 6b, the approximate solutions when $(\alpha, \beta, \gamma)^t = (0.5, 0.4, 0.3)^t$ and $(0.75, 0.5, 0.25)^t$ have been plotted, respectively.



(a) $\alpha = 1$. (b) $\alpha = 0.75$.

Figure 4. Plots of system (4.4) for Case 3.



(a) $\alpha = 1$. (b) $\alpha = 0.75$.

Figure 5. Plots of system (4.4) for Case 4.

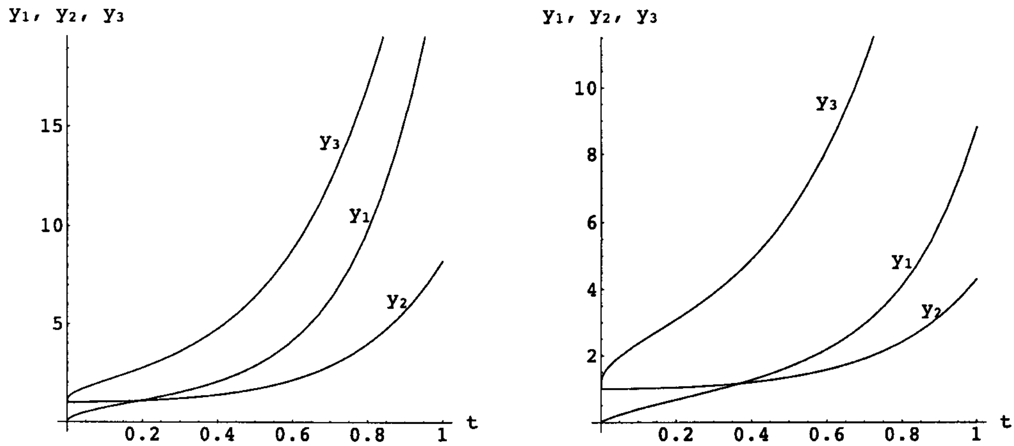
It should be noted that the approximate solutions plotted in Figures 6a and 6b have been obtained using seven terms of the decomposition series. It is evident that the accuracy can be improved by computing more terms of the approximate solution.

EXAMPLE 4.4. Consider the system of nonlinear fractional integro-differential equations

$$\begin{aligned} D_*^\alpha n_1(t) &= n_1 \left[K_1 - \gamma_1 n_2 - \int_{t-T_0}^{t} f_1(t-s) n_2(s)\, ds\right], \qquad K_1 > 0, \quad 0 < \alpha \leqslant 1, \\ D_*^\alpha n_2(t) &= n_2 \left[-K_1 + \gamma_2 n_1 - \int_{t-T_0}^{t} f_2(t-s) n_1(s)\, ds\right], \qquad K_2 > 0, \end{aligned} \tag{4.14}$$

subject to the initial conditions

$$n_1(0) = N_1, \qquad n_2(0) = N_2. \tag{4.15}$$



(a) $\alpha = 0.5$, $\beta = 0.4$, and $\gamma = 0.3$.

(b) $\alpha = 0.75$, $\beta = 0.5$, and $\gamma = 0.25$.

Figure 6. Plots of system (4.11).

The decomposition method will be extended for solving a system of four integral equations derived from (4.14). Let us consider

$$\begin{aligned} D_*^\alpha n_1(t) &= m_1(t) \Rightarrow n_1(t) = n_1(0) + J^\alpha(m_1(t)), \\ D_*^\alpha n_2(t) &= m_2(t) \Rightarrow n_2(t) = n_1(0) + J^\alpha(m_2(t)). \end{aligned} \tag{4.16}$$

So we have the following system of four integral equations:

$$\begin{aligned} n_1(t) &= n_1(0) + J^\alpha(m_1(t)), \\ m_1(t) &= n_1(t)\left[K_1 - \gamma_1 n_2(t) - \int_{t-T_0}^{t} f_1(t-s) n_2(s)\, ds\right], \\ n_2(t) &= n_2(0) + J^\alpha(m_2(t)), \\ m_2(t) &= n_2(t)\left[-K_2 + \gamma_2 n_1(t) + \int_{t-T_0}^{t} f_2(t-s) n_1(s)\, ds\right]. \end{aligned} \tag{4.17}$$

Substituting from (4.16) into (4.17):

$$\begin{aligned} n_1(t) &= N_1 + J_0^\alpha m_1(s)\, ds, \\ m_1(t) &= K_1\left(N_1 + \int_0^t m_1(s)\, ds\right) - n_1(t)\left(\gamma_1 n_2(t) - \int_{t-T_0}^{t} e^{-(t-s)} n_2(s)\, ds\right), \\ n_2(t) &= N_2 + J_0^\alpha m_2(s)\, ds, \\ m_2(t) &= K_2\left(N_2 + \int_0^t m_2(s)\, ds\right) + n_2(t)\left(\gamma_2 n_1(t) + \int_{t-T_0}^{t} e^{-(t-s)} n_1(s)\, ds\right). \end{aligned}$$

Using the alternate algorithm for computing the Adomian polynomials, the Adomian procedure would be as follows:

$$\begin{aligned} n_{1,0} &= N_1, \\ m_{1,0} &= K_1 N_1, \\ n_{2,0} &= N_2, \\ m_{2,0} &= K_2 N_2, \end{aligned}$$

$$
\begin{aligned}
n_{1,j+1} &= J_0^\alpha m_{1,j}(s)\,ds,\\
m_{1,j+1} &= K_1\int_0^t m_{1,j}(s)\,ds-\gamma_1\sum_{k=0}^{j} n_{1,k}(t)n_{2,j-k}(t)-\int_{t-T_0}^{t} e^{-(t-s)}\left(\sum_{k=0}^{j} n_{1,k}(t)n_{2,j-k}(t)\right)ds,\\
n_{2,j+1} &= J_0^\alpha m_{2,j}(s)\,ds,\\
m_{2,j+1} &= K_2\int_0^t m_{2,j}(s)\,ds-\gamma_2\sum_{k=0}^{j} n_{1,k}(t)n_{2,j-k}(t)+\int_{t-T_0}^{t} e^{-(t-s)}\left(\sum_{k=0}^{j} n_{1,k}(t)n_{2,j-k}(t)\right)ds.
\end{aligned}
$$

A three-term approximation for n_1 and n_2 is as follows:

$$
\begin{aligned}
n_1(t) &= N_1+\frac{K_1N_1t^\alpha}{\Gamma(1+\alpha)}+\frac{N_1t^\alpha\left(\left(-1+e^{-T_0}\right)N_2+4^{-\alpha}{K_1}^2\sqrt{\pi}t^\alpha/\Gamma(1/2+\alpha)-N_2\gamma_1\right)}{\Gamma(1+\alpha)},\\
n_2(t) &= N_2+\frac{K_2N_2t^\alpha}{\Gamma(1+\alpha)}+\frac{N_2t^\alpha\left(\left(1-e^{-T_0}\right)N_1+4^{-\alpha}{K_2}^2\sqrt{\pi}t^\alpha/\Gamma(1/2+\alpha)-N_1\gamma_2\right)}{\Gamma(1+\alpha)},
\end{aligned}
\tag{4.18}
$$

when $\alpha = 1$, then we have

$$
\begin{aligned}
n_1(t) &= N_1+N_1\left[K_1-\gamma_1N_2-N_2\left(1-e^{-T_0}\right)\right]t+0.5N_1K_1^2t^2,\\
n_2(t) &= N_2+N_2\left[K_2-\gamma_2N_1-N_1\left(1-e^{-T_0}\right)\right]t+0.5N_2K_2^2t^2,
\end{aligned}
\tag{4.19}
$$

which is the same solution given by Biazar [17].

5. CONCLUDING REMARKS

In this work, we demonstrate that the Adomian decomposition method is well suited for solving systems of linear and nonlinear fractional integro-differential equations. The Adomian decomposition technique requires less computational work than existing approaches while supplying quantitatively reliable results. It is also shown that the solutions of the fractional equations reduce to the solutions of the corresponding integer order equations.

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